Heat-kernel coefficients of the Laplace operator on the D-dimensional ball

M. Bordag[*]

Universität Leipzig, Institut für Theoretische Physik,
Augustusplatz 10, 04109 Leipzig, Germany

E. Elizalde[†]

CEAB, CSIC, Camí de Santa Bàrbara, 17300 Blanes,
and Departament d'ECM and IFAE, Facultat de Física,
Universitat de Barcelona, Av. Diagonal 647, 08028 Barcelona

K. Kirsten[‡]

Departament d'ECM, Facultat de Física,
Universitat de Barcelona, Av. Diagonal 647, 08028 Barcelona
Spain

February 1, 2008

Abstract

We present a very quick and powerful method for the calculation of heat-kernel coefficients. It makes use of rather common ideas, as integral representations of the spectral sum, Mellin transforms, non-trivial commutation of series and integrals and skilful analytic continuation of zeta functions on the complex plane. We apply our method to the case of the heat-kernel expansion of the Laplace operator on a D-dimensional ball with either Dirichlet, Neumann or, in general, Robin boundary conditions. The final formulas are quite simple. Using this case as an example, we illustrate in detail our scheme —which serves for the calculation of an (in principle) arbitrary number of heat-kernel coefficients in any situation when the basis functions are known. We provide a complete list of new results for the coefficients $B_3, ..., B_{10}$, corresponding to the D-dimensional ball with all the mentioned boundary conditions and $D = 3, 4, 5$.

[*]E-mail address: bordag@qft.physik.uni-leipzig.d400.de
[†]E-mail address: eli@zeta.ecm.ub.es
[‡]Alexander von Humboldt foundation fellow, E-mail address: klaus@zeta.ecm.ub.es

1   Introduction

An important issue for more than twenty years now has been to obtain explicitly the coefficients which appear in the short-time expansion of the heat-kernel $K(t)$ corresponding to a Laplacian-like operator on a D-dimensional manifold $\mathcal{M}$. In mathematics this interest stems, in particular, from the well-known connections that exist between the heat-equation and the Atiyah-Singer index theorem [1]. In physics, the importance of that expansion is notorious in different domains of quantum field theory, where it is commonly known as the (integrated) Schwinger-De Witt proper-time expansion [2, 3]. In this context, the heat-equation for an elliptic (in general pseudoelliptic) differential operator P and the corresponding zeta function $\zeta_P(s)$ has been realized to be a particularly useful tool for the determination of effective actions [4] and for the calculation of vacuum or Casimir energies [5] (a fundamental issue for understanding the vacuum structure of a quantum field theory). Here usually the derivative $\zeta_P'(0)$ of the zeta function [4] and its value at $s = -1/2$ (sometimes the principal part) are needed [5, 6].

In this paper we would like to exploit another property of the zeta function $\zeta_P(s)$ corresponding to an elliptic operator P, namely its well-known close connection with the heat-kernel expansion. In spite of the fact that almost everybody is aware of such connection, its actual use in the literature has remained very scarce till now. If the manifold $\mathcal{M}$ has a boundary $\partial\mathcal{M}$, the coefficients B_n in the short-time expansion have both a volume and a boundary part [7, 8]. It is usual to write this expansion in the form

$$K(t) \sim (4\pi t)^{-\frac{D}{2}} \sum_{k=0,1/2,1,\dots}^{\infty} B_k t^k, \tag{1.1}$$

with

$$B_k = \int_{\mathcal{M}} dV \ b_n + \int_{\partial\mathcal{M}} dS \ c_n. \tag{1.2}$$

For the volume part very effective systematic schemes have been developed (see for example [9, 10, 11]). The calculation of c_n, however, is in general more difficult. Only quite recently has the coefficient c_2 for Dirichlet and for Neumann boundary conditions been found [12-17]. Very new results on the coefficient $B_{5/2}$ for manifolds with totally geodesic boundaries will be given in [18].

When using the general formalism of Ref. [12] for higher-spin particles, Moss and Poletti [19, 20] found a discrepancy with the direct calculations of D'Eath and Esposito [21] (see also [22-25]). The latter results have been confirmed in [26, 27], where a new systematic scheme for the calculation of c_2 has been designed in the context of the Hartle-Hawking wave-function of the universe and for the case when the whole set of basis functions is known [26, 27]. Finally, very recently the discrepancy has been resolved completely [28] and now the results that are found using the general algorithm [29] are in agreement with those coming from the direct calculations [21-27].

The connection between the heat-kernel expansion, Eq. (1.1) and the associated zeta

function is established through the formulas [30]

$$Res \ \zeta(s) = \frac{B_{\frac{m}{2}-s}}{(4\pi)^{\frac{m}{2}}\Gamma(s)}, \tag{1.3}$$

for $s = \frac{m}{2}, \frac{m-1}{2}, ..., \frac{1}{2}; -\frac{2l+1}{2}$, for $l \in \mathbb{N}_0$, and

$$\zeta(-p) = (-1)^p p! \frac{B_{\frac{m}{2}+p}}{(4\pi)^{\frac{m}{2}}}, \tag{1.4}$$

for $p \in \mathbb{N}_0$. The aim of the present article is to show that these equations, (1.3) and (1.4), can actually serve as a very convenient starting point for the calculation of th coefficients B_k, even in the cases when the eigenvalues of the operator P under consideration are not known. The good knowledge in explicit zeta-function evaluations that have been accumulated in the past few years (for a review of many results in this respect, see [31]) will allow us to elaborate a very competitive method of calculation of the heat-kernel coefficients which makes use of rather common ingredients, such as integral representations of the spectral sum, Mellin transforms, non-trivial commutation of series and integrals and skilful analytic continuation of zeta functions on the complex plane.

To explain the method in detail we will consider the Laplace operator on the D-dimensional ball with Dirichlet, Neumann or (in general) Robin boundary conditions. Earlier investigations on the first few coefficients are due, for $D = 1$, to Stewartson and Waechter [32], to Waechter in $D = 2$ [33] and to Kennedy [34, 35] in up to $D = 5$ dimensions (for recent results on the functional determinant of the Laplace operator on the three and four dimensional ball see [36]). In these references the method was based on the use of Laplace transformations of the heat-kernel $K(t)$ itself. In that method an intermediate cut off has to be introduced at some point —because one needs to consider the Laplace transform of a function which is singular at $t = 0$. In contrast, in our approach it is the complex argument s of the zeta function of the Laplace operator which very neatly serves for the regularization of all sums (in just the usual way [31]).

The layout of the paper is as follows. In section 2 we briefly describe the eigenvalue problem of the massive Laplace operator on the ball and derive a representation of the associated zeta function in terms of a contour integral. We consider the *massive* Laplace operator because the analytical continuation procedure is slightly easier for the case of non-vanishing mass. In section 3 we describe how an analytical representation of the zeta function —valid in the strip $(1 - N)/2 < \Re s < 1$— can be obtained for any N, restricting our considerations in this section to $D = 3$ and to the case of Dirichlet boundary conditions. This representation will display very clearly the meromorphic structure of the zeta function. As is then shown in section 4, from this representation it is quite immediate to read off special properties, as the ones reflected by (1.3) and (1.4), in order to find the heat-kernel coefficients. In section 5 we explain the small changes in the procedure that are necessary in order to treat Robin boundary conditions, in general. Finally, in section 6 we study the modification to be introduced in the formulas for considering any arbitrary dimension D. In appendix A we exhibit some technical details of the calculation and

in Apps. B, C and D we give explicit tables of the heat-kernel coefficients for Dirichlet, Neumann and general Robin boundary conditions, for the dimensions $D = 3, 4, 5$.

2   Heat-kernel coefficients on the D-dimensional ball

As explained in the introduction, we are interested in the zeta function of the operator $(-\Delta + m^2)$ on the D-dimensional ball $B^D = \{x \in \mathbb{R}^D; |x| \leq R\}$ endowed with Dirichlet, Neumann or Robin boundary conditions. The zeta function is formally defined as

$$\zeta(s) = \sum_k \lambda_k^{-s}, \tag{2.1}$$

with the eigenvalues λ_k being determined through

$$(-\Delta + m^2)\phi_k(x) = \lambda_k \phi_k(x) \tag{2.2}$$

(k is in general a multiindex here), together with one of the three boundary conditions above. It is convenient to introduce a spherical coordinate basis, with $r = |x|$ and $D - 1$ angles $\Omega = (\theta_1, ..., \theta_{D-2}, \varphi)$. In these coordinates, a complete set of solutions of Eq. (2.2) together with one of the mentioned boundary conditions may be given in the form

$$\phi_{l,m,n}(r, \Omega) = r^{1-\frac{D}{2}} J_{l+\frac{D-2}{2}}(w_{l,n}r) Y_{l+\frac{D}{2}}(\Omega), \tag{2.3}$$

with $J_{l+(D-2)/2}$ being Bessel functions and $Y_{l+D/2}$ hyperspherical harmonics [37]. The $w_{l,n}$ (> 0) are determined through the boundary conditions by

$$J_{l+\frac{D-2}{2}}(w_{l,n}R) = 0, \quad \text{for Dirichlet boundary conditions,}$$

$$\frac{u}{R}J_{l+\frac{D-2}{2}}(w_{l,n}R) + w_{l,n} J'_{l+\frac{D-2}{2}}(w_{l,n}r)|_{r=R} = 0, \quad \text{for Robin boundary conditions.} \tag{2.4}$$

As is clear, the case $u = (1 - D/2)$ of the (general) Robin boundary conditions corresponds to the Neumann boundary conditions. In this notations, using $\lambda_{l,n} = w_{l,n}^2 + m^2$, the zeta function can be given in the form

$$\zeta(s) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} d_l(D)(w_{l,n}^2 + m^2)^{-s}, \tag{2.5}$$

where $w_{l,n}$ (> 0) is defined as the n-th root of the l-th equation. Here the sum over n is extended over all possible roots $w_{l,n}$ on the positive real axis, and $d_l(D)$ is the number of independent harmonic polynomials, which defines the degeneracy of each value of l and n in D dimensions. Explicitly,

$$d_l(D) = (2l + D - 2)\frac{(l + D - 3)!}{l!\,(D-2)!}. \tag{2.6}$$

Furthermore, here and in what follows the prime will always mean derivative of the function with respect to its argument.

4

To distinguish in the notation among the different cases, we will use the indices D, N and R to denote Dirichlet, Neumann and Robin boundary conditions, respectively. Thus, we will write ζ_D, ζ_N and ζ_R for the corresponding zeta functions. Using for the moment the unified notation $\Phi_{l+(D-2)/2}(w_{l,n}R) = 0$ for the boundary condition Eq. (2.4), it turns out that Eq. (2.5) may be written under the form of a contour integral on the complex plane,

$$\zeta(s) = \sum_{l=0}^{\infty} d_l(D) \int_{\gamma} \frac{dk}{2\pi i} \, (k^2 + m^2)^{-s} \frac{\partial}{\partial k} \ln \Phi_{l+\frac{D-2}{2}}(kR), \tag{2.7}$$

where the contour γ runs counterclockwise and must enclose all the solutions of (2.4) on the positive real axis (for a similar treatment of the zeta function as a contour integral see [38, 27, 39]). This representation of the zeta function in terms of a contour integral around some circuit γ on the complex plane, Eq. (2.7), is the *first step* of our procedure.

Depending on the value of the dimension D and on the boundary conditions chosen, the analysis of the zeta function, Eq. (2.7) —to be given below— will differ, but just in small details. For this reason, we will only describe at length the case of the three-dimensional ball with Dirichlet boundary condition. The derivation of the analogous results for the other boundary conditions and higher dimensions will then be clear, and shall be indicated only briefly.

3 A quick procedure for calculating heat-kernel coefficients

As explained above, we will illustrate the procedure in the case of the three-dimensional ball with Dirichlet boundary conditions. For $D = 3$ the degeneracy is $d_l(3) = 2l + 1$, so that the starting point of the calculation reads (we omit further indication of the dimension in the notation)

$$\zeta_D(s) = \sum_{l=0}^{\infty} (2l + 1) \int_{\gamma} \frac{dk}{2\pi i} \, (k^2 + m^2)^{-s} \frac{\partial}{\partial k} \ln J_{l+\frac{1}{2}}(kR). \tag{3.1}$$

As it stands, the representation (3.1) is valid for $\Re s > 3/2$. However, we are interested in the properties of $\zeta_D(s)$ in the range $\Re s < 0$ and thus, we need to perform the analytical continuation to the left domain of the complex plane. Before considering in detail the l-summation, we will first proceed with the k-integral alone.

The first specific idea is to shift the integration contour and place it along the imaginary axis. In order to avoid contributions coming from the origin $k = 0$, we will consider (with $\nu = l + 1/2$) the expression

$$\zeta_D^{\nu} = \int_{\gamma} \frac{dk}{2\pi i} \, (k^2 + m^2)^{-s} \frac{\partial}{\partial k} \ln \left(k^{-\nu} J_{\nu}(kR) \right), \tag{3.2}$$

where the additional factor $k^{-\nu}$ in the logarithm does not change the result, for no additional pole is enclosed. One then easily obtains

$$\zeta_D^\nu = \frac{\sin(\pi s)}{\pi} \int\limits_m^\infty dk \; [k^2 - m^2]^{-s} \frac{\partial}{\partial k} \ln\left(k^{-\nu} I_\nu(kR)\right) \tag{3.3}$$

valid in the strip $1/2 < \Re s < 1$. A similar representation valid for $m = 0$ has been given in [40, 41].

As the *second step* of our method, we make use of the uniform expansion of the Bessel function $I_\nu(k)$ for $\nu \to \infty$ as $z = k/\nu$ fixed [42]. One has

$$I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{\frac{1}{4}}} \left[1 + \sum_{k=1}^\infty \frac{u_k(t)}{\nu^k}\right], \tag{3.4}$$

with $t = 1/\sqrt{1+z^2}$ and $\eta = \sqrt{1+z^2} + \ln[z/(1+\sqrt{1+z^2})]$. The first few coefficients are listed in [42], higher coefficients are immediate to obtain by using the recursion [42]

$$u_{k+1}(t) = \frac{1}{2} t^2 (1-t^2) u_k'(t) + \frac{1}{8} \int\limits_0^t d\tau \; (1 - 5\tau^2) u_k(\tau), \tag{3.5}$$

starting with $u_0(t) = 1$. As is clear, all the $u_k(t)$ are polynomials in t. Furthermore, the coefficients $D_n(t)$ defined by

$$\ln\left[1 + \sum_{k=1}^\infty \frac{u_k(t)}{\nu^k}\right] \sim \sum_{n=1}^\infty \frac{D_n(t)}{\nu^n} \tag{3.6}$$

are easily found with the help of a simple computer program.

Now comes what can be considered as the *third step* of our method. By adding and subtracting N leading terms of the asymptotic expansion, Eq. (3.6), for $\nu \to \infty$, Eq. (3.3) may be split into the following pieces

$$\zeta_D^\nu = Z_D^\nu(s) + \sum_{i=-1}^N A_i^{\nu,D}(s), \tag{3.7}$$

with the definitions

$$Z_D^\nu(s) = \frac{\sin(\pi s)}{\pi} \int\limits_{mR/\nu}^\infty dz \; \left[\left(\frac{z\nu}{R}\right)^2 - m^2\right]^{-s} \frac{\partial}{\partial z} \left\{\ln\left[z^{-\nu} I_\nu(z\nu)\right]\right. \tag{3.8}$$

$$\left. - \ln\left[\frac{z^{-\nu}}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{\frac{1}{4}}}\right] - \sum_{n=1}^N \frac{D_n(t)}{\nu^n}\right\},$$

and

$$A_{-1}^{\nu,D} = \frac{\sin(\pi s)}{\pi} \int\limits_{mR/\nu}^\infty dz \; \left[\left(\frac{z\nu}{R}\right)^2 - m^2\right]^{-s} \frac{\partial}{\partial z} \ln\left(z^{-\nu} e^{\nu\eta}\right), \tag{3.9}$$

6

$$A_0^{\nu,D} \;=\; \frac{\sin(\pi s)}{\pi} \int\limits_{mR/\nu}^{\infty} dz \left[\left(\frac{z\nu}{R}\right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \ln(1+z^2)^{-\frac{1}{4}}, \tag{3.10}$$

$$A_i^{\nu,D} \;=\; \frac{\sin(\pi s)}{\pi} \int\limits_{mR/\nu}^{\infty} dz \left[\left(\frac{z\nu}{R}\right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \left(\frac{D_i(t)}{\nu^i}\right). \tag{3.11}$$

The essential idea is conveyed here by the fact that the representation (3.7) has the following important properties. First, by considering the asymptotics of the integrand in Eq. (3.8) for $z \to mR/\nu$ and $z \to \infty$, it can be seen that the function

$$Z_D(s) = \sum_{l=0}^{\infty} (2l+1) Z_D^{l+\frac{1}{2}}(s)$$

is analytic on the strip $(1-N)/2 < \Re s < 1$. For this reason, it gives no contribution to the residue of $\zeta_D(s)$ in that strip. Furthermore, for $s = -k$, $k \in \mathbb{N}_0$, $k < -1 + N/2$, we have $Z(s) = 0$ and, thus, it also yields no contribution to the values of the zeta function at these points. Together with Eqs. (1.3) and (1.4), this result means that the heat-kernel coefficients are just determined by the terms $A_i^D(s)$ with

$$A_i^D(s) = \sum_{l=0}^{\infty} (2l+1) A_i^{l+\frac{1}{2},D}(s). \tag{3.12}$$

As they stand, the $A_i^{\nu,D}(s)$ in Eqs. (3.9), (3.10) and (3.11) are well defined on the strip $1/2 < \Re s < 1$ (at least). And we will now show that the analytic continuation in the parameter s to the whole of the complex plane, in terms of known functions, can be performed. Keeping in mind that $D_i(t)$ is a polynomial in t, all the $A_i^{\nu,D}(s)$ are in fact hypergeometric functions, which is seen by means of the basic relation [43]

$$_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}.$$

Let us consider first in detail $A_{-1}^{\nu,D}(s)$, , and the corresponding $A_{-1}^D(s)$, $A_0^D(s)$. One finds immediately that

$$A_{-1}^{\nu,D}(s) \;=\; \frac{m^{-2s}}{2\sqrt{\pi}} Rm \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\; _2F_1\left(-\frac{1}{2}, s-\frac{1}{2}; \frac{1}{2}; -\left(\frac{\nu}{mR}\right)^2\right) - \frac{\nu}{2} m^{-2s}, \tag{3.13}$$

$$=\; -\frac{1}{4} m^{-2s}\; _2F_1\left(1,s;1;-\left(\frac{\nu}{mR}\right)^2\right) = -\frac{1}{4} m^{-2s}\left[1+\left(\frac{\nu}{mR}\right)^2\right]^{-s}, \tag{3.14}$$

where in the last equality we have used that $_2F_1(a,s;a;x) = (1-x)^{-s}$.

The next step is to consider the summation over l. For $A_{-1}^{\nu,D}(s)$ this is best done using a Mellin-Barnes type integral representation of the hypergeometric functions

$$_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_C dt\, \frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(c+t)} (-z)^t, \tag{3.15}$$

7

where the contour is such that the poles of $\Gamma(a + t)\Gamma(b + t)/\Gamma(c + t)$ lie to the left of it and the poles of $\Gamma(-t)$ to the right [43]. After interchanging the summation over l and the integration in (3.15), the result will be a Hurwitz zeta function, which is defined as

$$\zeta_H(s; v) = \sum_{l=0}^{\infty} (l + v)^{-s}, \qquad \Re s > 1. \tag{3.16}$$

However, as is well known, one has to be very careful with this kind of manipulations, what has been realized and explained with great detail in [44, 45, 46]. This point is of crucial importance (it has been the source of many errors in the literature over the past ten years [31]) and can be considered as the *fourth step* of our original procedure here. Applying the method, as described in the mentioned references, to $A_{-1}^D(s)$,

$$A_{-1}^D(s) = \sum_{l=0}^{\infty} (2l + 1) \left[\frac{m^{-2s}}{2\sqrt{\pi}} Rm \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \, {}_2F_1\left(-\frac{1}{2}, s - \frac{1}{2}; \frac{1}{2}; -\left(\frac{l + \frac{1}{2}}{mR}\right)^2\right) \right. $$
$$\left. - \frac{l + \frac{1}{2}}{2} m^{-2s} \right],$$

it turns out that we may interchange the $\sum_l$ and the integral in Eq. (3.15) *only* if for the real part $\Re C$ of the contour the condition $\Re C < -1$ is satisfied. However, the argument $\Gamma(-1/2 + t)\Gamma(s - 1/2)/\Gamma(1/2 + t)$ has a pole at $t = 1/2$. Thus the contour $\mathcal{C}$ coming from $-i\infty$ must cross the real axis to the right of $t = 1/2$, and then once more between 0 and $1/2$ (in order that the pole $t = 0$ of $\Gamma(-t)$ lies to the right of it), before going to $+i\infty$. That is, before interchanging the sum and the integral we have to shift the contour $\mathcal{C}$ over the pole at $t = 1/2$ to the left, cancelling the (potentially divergent) second piece in $A_{-1}^D(s)$. Closing then the contour to the left, we end up with the following expression in terms of Hurwitz zeta functions

$$A_{-1}^D(s) = \frac{R^{2s}}{2\sqrt{\pi}\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \frac{\Gamma\left(j + s - \frac{1}{2}\right)}{s + j} \zeta_H(2j + 2s - 2; 1/2). \tag{3.17}$$

For A_0^D one only needs to use the binomial expansion in order to find

$$A_0^D(s) = -\frac{R^{2s}}{2\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \Gamma(s + j) \zeta_H(2j + 2s - 1; 1/2). \tag{3.18}$$

The series are convergent for $|mR| < 1/2$. These representations (3.17) and (3.18) show very clearly the analytic structure of $A_{-1}^D(s)$ and $A_0^D(s)$. As the *fifth* (and final) *step* of our procedure, we are left with the quite simple task of explictly evaluating this analytic structure, namely of finding its poles and some point values, and of adding all contributions together.

The point values $A_{-1,0}^D(-p)$, $p \in \mathbb{N}_0$ —respectively their residues in $s = 1/2, -(2l + 1)/2$, $l \in \mathbb{N}_0$— necessary for the calculation of the associated heat-kernel coefficients are

immediate to obtain, using just

$$\zeta_H(1 + \epsilon, 1/2) = \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0),$$

$$\Gamma(\epsilon - n) = \frac{1}{\epsilon} \frac{(-1)^n}{n!} + \mathcal{O}(\epsilon^0). \tag{3.19}$$

However, before we can actually calculate (an in principle arbitrary number of) the heat-kernel coefficients, we need to obtain analytic expressions for the $A_i^D(s)$, $i \in \mathbb{N}$. As is easy to see, they are similar to the ones for $A_{-1}^D(s)$ and $A_0^D(s)$ above. We need to recall only that $D_i(t)$, Eq. (3.6), is a polynomial in t,

$$D_i(t) = \sum_{a=0}^{i} x_{i,a} t^{i+2a}, \tag{3.20}$$

which coefficients $x_{i,a}$ are easily found by using Eqs. (3.5) and (3.6) directly, or either by using the direct recursion relations presented in appendix A. Thus the calculation of $A_i^{\nu,D}(s)$ is essentially solved through the identity

$$\int_{mR/\nu}^{\infty} dz \left[\left(\frac{z\nu}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} t^n = -m^{-2s} \frac{n}{2(mR)^n} \frac{\Gamma\left(s + \frac{n}{2}\right) \Gamma(1 - s)}{\Gamma\left(1 + \frac{n}{2}\right)}$$

$$\times \nu^n \left[1 + \left(\frac{\nu}{mR} \right)^2 \right]^{-s - \frac{n}{2}}. \tag{3.21}$$

The remaining sum may be done as mentioned for A_0^D, and we end up with

$$A_i^D(s) = -\frac{R^{2s}}{\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \zeta_H(-1 + i + 2j + 2s; 1/2)$$

$$\times \sum_{a=0}^{i} x_{i,a} \frac{(i + 2a)\Gamma\left(s + a + j + \frac{i}{2}\right)}{\Gamma\left(1 + a + \frac{i}{2}\right)}. \tag{3.22}$$

In summary we have obtained the analytic expression of all the asymptotic terms coming from expansion (3.4) in its most elementary form, which involves the very familiar Hurwitz zeta functions and Gamma functions only. Expressions (3.17), (3.18) and (3.22) constitute the explicit starting point for the calculation of an —in principle arbitrary— number of heat-kernel coefficients in an extremely quick way.

4 Heat-kernel coefficients for Dirichlet boundary conditions on the three-dimensional ball

Let us now see how the analysis in Sect. 2 can be used for a very effective calculation of the heat-kernel coefficients. The dependence of the coefficients on the mass is already contained in the coefficients of the massless case through

$$K_m(t) = K_{m=0}(t) e^{-m^2 t}$$

and for this reason we shall restrict ourselves to $m = 0$. For the sums in (3.17), (3.18) and (3.22) this means that only $j = 0$ will contribute.

We shall distinguish between the coefficients B_k with integer and half-integer index k, because the situation is actually different in both cases. In fact, corresponding to Eq. (1.3) (resp. Eq. (1.4)), the residue of (resp. the value of the function) ζ_D is needed.

Let us start with the case of integer index $k \in \mathbb{N}$, so that Res $\zeta_D(3/2 - k)$ is to be calculated. In order that $Z_D(s)$ does not contribute, one has to choose $N = 2k - 1$ and thus only the asymptotic terms $A_j^D(s)$, $j = -1, 0, 1, ..., 2k - 1$, will provide some contribution. Furthermore, one may see very easily which terms in the different $A_j^D(s)$ contribute. An important feature is, that for $i = 2n$, $n \in \mathbb{N}_0$, $A_i^D(s)$ does not contribute to B_k for $k \in \mathbb{N}$. The relevant residues are found to be

$$\text{Res } A_{-1}^D \left(\frac{3}{2} - k \right) = \frac{(-1)^{k-1}}{(k-1)!} \frac{R^{3-2k}}{2\sqrt{\pi}\Gamma\left(\frac{5}{2} - k\right)} \zeta_H\left(1 - 2k; \frac{1}{2}\right),$$

$$\text{Res } A_{2k-1}^D \left(\frac{3}{2} - k \right) = -\frac{R^{3-2k}}{2\Gamma\left(\frac{3}{2} - k\right)} \sum_{a=0}^{2k-1} x_{2k-1,a} \frac{(2k - 1 + 2a)a!}{\Gamma\left(\frac{1}{2} + a + k\right)},$$

and for $n \in \mathbb{N}$, $n \leq k - 1$, $k \leq 3n$,

$$\text{Res } A_{2n-1} \left(\frac{3}{2} - k \right) = \frac{(-1)^k R^{3-2k}}{\Gamma\left(\frac{3}{2} - k\right)} \zeta_H\left(1 + 2n - 2k; \frac{1}{2}\right) \sum_{a=0}^{k-1-n} x_{2n-1,a} \frac{(-1)^{a+n}(2n + 2a - 1)}{(k - 1 - a - n)!},$$

whereas for $n \leq k - 1$, $k > 3n$, we have

$$\text{Res } A_{2n-1} \left(\frac{3}{2} - k \right) = \frac{(-1)^k R^{3-2k}}{\Gamma\left(\frac{3}{2} - k\right)} \zeta_H\left(1 + 2n - 2k; \frac{1}{2}\right) \sum_{a=0}^{2n-1} x_{2n-1,a} \frac{(-1)^{a+n}(2n + 2a - 1)}{(k - 1 - a - n)!}.$$

From these results we readily obtain the heat-kernel coefficients through

$$\text{Res } \zeta_D \left(\frac{3}{2} - k \right) = \text{Res } \sum_{l=0}^{k} A_{2l-1}^D \left(\frac{3}{2} - k \right) \equiv \frac{B_k}{(4\pi)^{\frac{3}{2}}\Gamma\left(\frac{3}{2} - k\right)}.$$

The coefficients up to B_{10} are listed in appendix B.

Let us now consider the calculation of the coefficients corresponding to half-integer index $B_{k+1/2}$, $k \in \mathbb{N}$. Here the value of $\zeta_D(3/2 - k)$ is needed and one finds $N = 2k$. It is apparent that the $A_i^D(s)$ with odd i, $i = 2j - 1$, $j \in \mathbb{N}_0$, do not contribute now. The relevant values of the $A_i^D(s)$ read

$$A_0^D(1 - k) = -\frac{R^{2-2k}}{2} \zeta_H\left(1 - 2k; \frac{1}{2}\right),$$

$$A_{2k}^D(1 - k) = (-1)^k(k - 1)! R^{2-2k} \sum_{a=0}^{2k} x_{2k,a} \frac{a!}{(a + k - 1)!},$$

and for $n \in \mathbb{N}$, $n \le k - 1$, $k \le 3n - 1$,

$$A_{2n}^D(1 - k) = -2R^{2-2k}(k - 1)!\zeta_H\left(1 + 2n - 2k; \frac{1}{2}\right)\sum_{a=0}^{k-n-1} x_{2n,a}\frac{(-1)^{n+a}}{(k - n - a - 1)!(a + n - 1)!},$$

whereas for $n \le k - 1$, $k > 3n - 1$, we have

$$A_{2n}^D(1 - k) = -2R^{2-2k}(k - 1)!\zeta_H\left(1 + 2n - 2k; \frac{1}{2}\right)\sum_{a=0}^{2n} x_{2n,a}\frac{(-1)^{n+a}}{(k - n - a - 1)!(a + n - 1)!}.$$

And from these results, we finally obtain

$$\zeta_D(1 - k) = \sum_{n=0}^{k} A_{2n}(1 - k) \equiv \frac{(-1)^{k-1}(k - 1)!}{(4\pi)^{\frac{3}{2}}}B_{k+\frac{1}{2}}.$$

The heat-kernel coefficients $B_{k+\frac{1}{2}}$ are listed in appendix B too. Using $\zeta_H(-n; q) = -B_{n+1}(q)/(n + 1)$, $n \in \mathbb{N}_0$, the results might have been given, equivalently, in terms of Bernoulli polynomials $B_{n+1}(q)$.

5 Robin boundary conditions on the three-dimensional ball

When Robin boundary conditions are imposed, using the same method of the preceding sections we can write the zeta function as

$$\zeta_R(s) = \sum_{l=0}^{\infty}(2l + 1)\int_\gamma \frac{dk}{2\pi i}\,(k^2 + m^2)^{-s}\frac{\partial}{\partial k}\ln\left[\frac{u}{R}J_{l+\frac{1}{2}}(kR) + kJ'_{l+\frac{1}{2}}(kR)\right] \qquad (5.1)$$

and, in analogy with Eq. (3.3), we then consider

$$\zeta_R^\nu = \frac{\sin(\pi s)}{\pi}\int_m^\infty dk\,[k^2 - m^2]^{-s}\frac{\partial}{\partial k}\ln\left[k^{-\nu}(\frac{u}{R}I_\nu(kR) + kI'_\nu(kR))\right]. \qquad (5.2)$$

Employing the same idea as for Dirichlet boundary conditions, this time we have in addition the following uniform asymptotic expansion [42]

$$I'_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}}\frac{e^{\nu\eta}(1 + z^2)^{\frac{1}{4}}}{z}\left[1 + \sum_{k=1}^{\infty}\frac{v_k(t)}{\nu^k}\right], \qquad (5.3)$$

with the $v_k(t)$ determined by

$$v_k(t) = u_k(t) + t(t^2 - 1)\left[\frac{1}{2}u_{k-1}(t) + tu'_{k-1}(t)\right].$$

11

In analogy with Eq. (3.6), we write

$$\ln\left[1 + \sum_{k=1}^{\infty}\frac{v_k(t)}{\nu^k} + \frac{u}{\nu}t\left(1 + \sum_{k=1}^{\infty}\frac{u_k(t)}{\nu^k}\right)\right] \sim \sum_{n=1}^{\infty}\frac{M_n(t)}{\nu^n}, \tag{5.4}$$

where the functions $M_n(t)$ are easily obtained. At this point we see already, that for Robin boundary conditions no additional calculation is necessary. Comparing the expansion (5.3) with (3.4) and introducing $A_i^R(s)$ for the contributions coming from the asymptotic terms, one has

$$A_{-1}^R(s) = A_{-1}^D(s), \qquad A_0^R(s) = -A_0^D(s). \tag{5.5}$$

Furthermore, the functions $M_i(t)$ are of the form

$$M_i(t) = \sum_{a=0}^{2i} z_{i,a} t^{i+a} \tag{5.6}$$

(notice that here, in contrast with the case of Dirichlet boundary conditions, all powers between i and $3i$ are present). As a result, we find

$$\begin{aligned}
A_i^M(s) &= -\frac{R^{2s}}{\Gamma(s)}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(mR)^{2j}\zeta_H(-1+i+2j+2s;1/2) \\
&\qquad\times\sum_{a=0}^{2i} z_{i,a}\frac{(i+a)\Gamma\left(s+j+\frac{i+a}{2}\right)}{\Gamma\left(1+\frac{i+a}{2}\right)}.
\end{aligned} \tag{5.7}$$

One can show again that only the even indices i contribute to the residues of $\zeta_R(s)$, whereas the odd ones will contribute to the point values.

Restricting ourselves as before (see the comment in the previous section) to the massless case, the results for the heat-kernel coefficients may now be read off from the formulas in the previous section. One has

$$\begin{aligned}
\text{Res } A_{-1}^R\left(\frac{3}{2}-k\right) &= \text{Res } A_{-1}^D\left(\frac{3}{2}-k\right), \\
\text{Res } A_{2k-1}^R\left(\frac{3}{2}-k\right) &= -\frac{R^{3-2k}}{2\Gamma\left(\frac{3}{2}-k\right)}\sum_{a=0}^{4k-2} z_{2k-1,a}\frac{(2k-1+a)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(\frac{1}{2}+k+\frac{a}{2}\right)},
\end{aligned}$$

where the expressions for A_{2n-1}^R are found from the results in Sect. 3, once $x_{2n-1,a}$ has been replaced with $z_{2n-1,2a}$. The coefficients for Neumann boundary conditions are given in appendix C, and for the general case (u arbitrary) in appendix D. For the point values the analogous formulas read

$$\begin{aligned}
A_0^R(1-k) &= -A_0^D(1-k), \\
A_{2k}^R(1-k) &= (-1)^k R^{2-2k}(k-1)!\sum_{a=0}^{4k} z_{2k,a}\frac{(2k+a)\Gamma\left(1+\frac{a}{2}\right)}{2\Gamma\left(1+k+\frac{a}{2}\right)},
\end{aligned}$$

and once more the replacement of $x_{2n,a}$ with $z_{2n,2a}$ leads to the results for A_{2n}^R. The results for the heat-kernel coefficients are summarized in Apps. C and D.

12

6   Generalization to the D-dimensional ball

As we will now explain, for the generalization of our results to the case of a D-dimensional ball almost no additional calculations are necessary. Let us discuss first the case of Dirichlet boundary condition. The starting point of the analysis is now

$$\zeta_D(s) = \sum_{l=0}^{\infty} d_l(D) \int_{\gamma} \frac{dk}{2\pi i} \ (k^2 + m^2)^{-s} \frac{\partial}{\partial k} \ln J_{l+\frac{D-2}{2}}(kR). \tag{6.8}$$

It is easy to see that the above treatment for the individual terms of the l-series,

$$\zeta_D^\nu = \int_{\gamma} \frac{dk}{2\pi i} \ (k^2 + m^2)^{-s} \frac{\partial}{\partial k} \ln J_\nu(kR) \tag{6.9}$$

remains valid, once we have set $\nu = l + (D-2)/2$. In order to use our procedure for the whole l-summation, what remains to be done is to substitute for the degeneracy $d_l(D)$ its value in powers of $l + (D-2)/2$, in order to find again expressions in terms of the Hurwitz zeta function $\zeta_H(s; (D-2)/2)$. Writing

$$d_l(D) = \sum_{\alpha=1}^{D-2} e_\alpha(D) \left(l + \frac{D-2}{2}\right)^\alpha, \tag{6.10}$$

the final results for $A_{-1}^D(s)$, $A_0^D(s)$ and $A_i^D(s)$, $i \in \mathbb{N}$, may be read off from Eqs. (3.17), (3.18) and (3.22). We find

$$
\begin{aligned}
A_{-1}^D(s) &= \frac{R^{2s}}{4\sqrt{\pi}\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \frac{\Gamma\left(j + s - \frac{1}{2}\right)}{s+j} \\
&\quad \times \left[\sum_{\alpha=1}^{D-2} e_\alpha(D)\zeta_H(2j + 2s - 1 - \alpha; (D-2)/2)\right],
\end{aligned} \tag{6.11}
$$

$$
\begin{aligned}
A_0^D(s) &= -\frac{R^{2s}}{4\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \Gamma(s+j) \\
&\quad \times \left[\sum_{\alpha=1}^{D-2} e_\alpha(D)\zeta_H(2j + 2s - \alpha; (D-2)/2)\right],
\end{aligned} \tag{6.12}
$$

$$
\begin{aligned}
A_i^D(s) &= -\frac{R^{2s}}{2\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \\
&\quad \times \left[\sum_{\alpha=1}^{D-2} e_\alpha(D)\zeta_H(-\alpha + i + 2j + 2s; (D-2)/2)\right] \\
&\quad \times \sum_{a=0}^{i} x_a \frac{(i + 2a)\Gamma\left(s + a + j + \frac{i}{2}\right)}{\Gamma\left(1 + a + \frac{i}{2}\right)}.
\end{aligned} \tag{6.13}
$$

We shall spare the reader the analogous results for Robin boundary conditions. They need not be given explicitly, since the procedure is absolutely clear by now. Let us just

13

write down the relevant residues and point values of $\zeta_D(s)$ (the Robin case follows from the replacements explained in Sect. 5). They read

$$\text{Res } A^D_{-1}\left(\frac{3}{2}-k\right) = \frac{(-1)^{k-1}}{(k-1)!}\frac{R^{3-2k}}{4\sqrt{\pi}\Gamma\left(\frac{5}{2}-k\right)}\sum_{\alpha=1}^{D-2} e_\alpha(D)\,\zeta_H\left(2-2k-\alpha;\frac{D-2}{2}\right),$$

for $n = 1, ..., k-1$, $k > 3n$,

$$\text{Res } A_{2n-1}\left(\frac{3}{2}-k\right) = (-1)^k\frac{R^{3-2k}}{2\Gamma\left(\frac{3}{2}-k\right)}\sum_{\alpha=1}^{D-2} e_\alpha(D)\,\zeta_H\left(2+2n-\alpha-2k;\frac{D-2}{2}\right)$$

$$\times \sum_{a=0}^{2n-1}(-1)^{n+a}x_{2n-1,a}\frac{(2n+2a-1)}{(k-1-n-a)!\Gamma\left(\frac{1}{2}+a+n\right)},$$

whereas for $k \leq n$, it reads

$$\text{Res } A_{2n-1}\left(\frac{3}{2}-k\right) = (-1)^k\frac{R^{3-2k}}{2\Gamma\left(\frac{3}{2}-k\right)}\sum_{\alpha=1}^{D-2} e_\alpha(D)\,\zeta_H\left(2+2n-\alpha-2k;\frac{D-2}{2}\right)$$

$$\times \sum_{a=0}^{k-n-1}(-1)^{n+a}x_{2n-1,a}\frac{(2n+2a-1)}{(k-1-n-a)!\Gamma\left(\frac{1}{2}+a+n\right)}.$$

For higher indices it is adviceable to distinguish between D even and D odd. For D odd contributions arise for $n = k, ..., k+(D-3)/2$, and read

$$\text{Res } A_{2n-1}\left(\frac{3}{2}-k\right) = -\frac{R^{3-2k}}{4\Gamma\left(\frac{3}{2}-k\right)}e_{1+2n-2k}\sum_{a=0}^{2n-1}x_{2n-1,a}\frac{(2n+2a-1)(a+n-k)!}{\Gamma\left(\frac{1}{2}+a+n\right)},$$

whereas for D even the indices run from $n = k, ..., k+(D-4)/2$, and the results are

$$\text{Res } A_{2n}\left(\frac{3}{2}-k\right) = -\frac{R^{3-2k}}{2\Gamma\left(\frac{3}{2}-k\right)}e_{2+2n-2k}\sum_{a=0}^{2n}x_{2n,a}\frac{\Gamma\left(\frac{3}{2}-k+a+n\right)}{(a+n-1)!}.$$

Let us conclude with the list of point values. The leading asymptotics A^D_{-1} gives only contributions for $k = 0$,

$$A_{-1}(0) = -\frac{1}{2}\sum_{\alpha=1}^{D-2} e_\alpha(D)\zeta_H\left(-\alpha-1;\frac{D-2}{2}\right).$$

Furthermore, for $n = 1, ..., k-1$, we have

$$A_{2n}\left(1-k\right) = -R^{2-2k}(k-1)!\sum_{\alpha=1}^{D-2} e_\alpha(D)\zeta_H\left(-\alpha+2n+2-2k;\frac{D-2}{2}\right)$$

$$\times \sum_{a=0}^{2n}x_{2n,a}\frac{(-1)^{a+n}}{(a+n-1)!(k-1-a-n)!},$$

14

if $k > 3n - 1$, and if $k \leq 3n - 1$

$$A_{2n}\left(1-k\right) = -R^{2-2k}(k-1)! \sum_{\alpha=1}^{D-2} e_\alpha(D)\, \zeta_H\left(-\alpha + 2n + 2 - 2k; \frac{D-2}{2}\right)$$

$$\times \sum_{a=0}^{k-n-1} x_{2n,a} \frac{(-1)^{a+n}}{(a+n-1)!(k-1-a-n)!}.$$

Finally, for D odd and for $n = k, ..., k + (D-3)/2$,

$$A_{2n}\left(1-k\right) = \frac{1}{2}(-1)^k(k-1)! R^{2-2k} e_{1+2n-2k} \sum_{a=0}^{2n} x_{2n,a} \frac{(a+n-k)!}{(a+n-1)!},$$

whereas for D even the indices run from $n = k+1, ..., k+(D-2)/2$, and the result reads

$$A_{2n-1}\left(1-k\right) = \frac{1}{4}(-1)^k(k-1)! R^{2-2k} e_{2n-2k} \sum_{a=0}^{2n-1} x_{2n-1,a} \frac{(2n-1+2a)\Gamma\left(\frac{1}{2}-k+a+n\right)}{\Gamma\left(\frac{1}{2}+a+n\right)}.$$

The formulas above simplify a bit if we write the degeneracy (2.6) under the form

$$d_l(D) = \frac{2}{(D-2)!}\left[\left(l+\frac{D-2}{2}\right)^2 - \left(\frac{D}{2}-2\right)^2\right] \times ... \times \left(l+\frac{D-2}{2}\right), \quad \text{for } D \text{ odd},$$

$$d_l(D) = \frac{2}{(D-2)!}\left[\left(l+\frac{D-2}{2}\right)^2 - \left(\frac{D}{2}-2\right)^2\right] \times ... \times \left(l+\frac{D-2}{2}\right)^2, \quad \text{for } D \text{ even},$$

so that $e_{2k}(D) = 0$ for D odd and $e_{2k-1}(D) = 0$ for D even, $k \in \mathbb{N}$.

Furthermore, one might use the following recursion for the coefficients $e_\alpha(D)$ appearing in the expression of the degeneracy $d_l(D)$, Eq. (6.10),

$$e_{2\alpha}(D+2) = \frac{1}{D(D-1)}\left[e_{2\alpha-2}(D) - \left(\frac{D}{2}-1\right)^2 e_{2\alpha}(D)\right], \quad \text{for } D \text{ even},$$

$$e_{2\alpha-1}(D+2) = \frac{1}{D(D-1)}\left[e_{2\alpha-3}(D) - \left(\frac{D}{2}-1\right)^2 e_{2\alpha-1}(D)\right], \quad \text{for } D \text{ odd},$$

where we have used the definitions $e_{-k}(D) = 0$ for $k \in \mathbb{N}_0$ and $e_\alpha(D) = 0$ for $\alpha > D - 2$.

We have performed explicit calculations for $D = 4$ and $D = 5$. One has in these cases

$$d_l(4) = (l+1)^2, \quad e_1(4) = 0, \; e_2(4) = 1,$$

$$d_l(5) = \frac{1}{3}\left(l+\frac{3}{2}\right)\left[\left(l+\frac{3}{2}\right)^2 - \frac{1}{4}\right], \quad e_1(5) = -\frac{1}{12}, \; e_2(5) = 0, \; e_3(5) = \frac{1}{3}.$$

The results for the heat-kernel coefficients are presented in Apps. B, C and D.

7   Conclusions

As promised in the introduction, we have developed in this paper a very convenient method in order to deal with the problem of the calculation of heat-kernel coefficients corresponding to an arbitrary elliptic operator with any of the usual boundary conditions (Dirichlet, Neumann or Robin), with the only proviso that the behavior of some basis for its spectrum should be known (even if the eigenvalues themselves are actually unknown).

This is indeed a very common case in mathematical physics, what conferes to our procedure a wide generality of application. Another fundamental characteristic of the method is its extreme simplicity, which comes in part from the quite strong background on zeta function computations that we have acquired during the last half a dozen years. This knowledge conferes to the new method the same elegance that the procedure of zeta function regularization (including the analytic continuation techniques and non-trivial series commutation that it involves) has in itself.

Finally, we have tried our method with explicit examples and give several tables of heat-kernel coefficients that have been calculated here (with relative easiness) for the first time. For the near future we envisage to investigate other physical applications where the method can prove useful.

Note: At the final stage of our analysis, P. Gilkey made us aware of related research by M. Levitin [47], who has further developed the approach of Kennedy [34, 35], also with the aim of calculating higher-order heat-kernel coefficients. We are indebted with M. Levitin for sending us his results, which have served as a very good check of our calculations. All results in common with his are in complete agreement.

A Recursion relation for the coefficients $x_{i,a}$

In this appendix we present the recursion relations for the coefficients $x_{i,a}$, Eq. (3.20). For convenience let us introduce for $i \in \mathbb{N}$, $a = 0, ..., i$,

$$x_{i,a} = \frac{c_{i+1,a}}{2^{i+1}(i + 2a)}.$$

Then, starting with $c_{1,0} = -1$, we find the following recursion relation,

$$c_{i,0} = (i - 2)c_{i-1,0} - \frac{1}{2}\sum_{s=1}^{i-1} c_{i-s,0}c_{s,0},$$

$$c_{i,i-1} = (4 - 3i)c_{i-1,i-2} + \frac{1}{2}\sum_{s=1}^{i-1} c_{i-s,i-s-1}c_{s,s-1},$$

and for $a = 1, ..., i - 2$, we have

$$
\begin{aligned}
c_{i,a} \;=\; & (i - 2 + 2a)(c_{i-1,a} - c_{i-1,a-1}) \\
& -\frac{1}{2}\sum_{s=1}^{i-1}\left(\sum_{j=Max(0,1+a+s-i)}^{Min(a,s-1)} c_{i-s,a-j}c_{s,j} - \sum_{j=Max(0,a+s-i)}^{Min(a-1,s-1)} c_{i-s,a-j-1}c_{s,j}\right).
\end{aligned}
$$

This relation can be used very effectively for the calculation of the coefficients $x_{i,a}$.

B Heat-kernel coefficients for Dirichlet boundary conditions

In this appendix we list our results for the heat-kernel coefficients of the Laplace operator in $3, 4$ and 5 dimensions with Dirichlet boundary conditions. Here and in the following appendices, the first coefficients $B_0, ..., B_{5/2}$ are listed for completeness and may also be found in [34, 35] or derived from [12].

In three dimensions we have found that

$$
\begin{aligned}
B_0 &= \frac{4}{3}\pi R^3 \\
B_{1/2} &= -2\pi^{3/2}R^2 \\
B_1 &= \frac{8\pi R}{3} \\
B_{3/2} &= -\frac{1}{6}\pi^{3/2} \\
B_2 &= -\frac{16\pi}{315R} \\
B_{5/2} &= -\frac{\pi^{3/2}}{120R^2}
\end{aligned}
$$

$$B_3 = -\frac{64\pi}{9009R^3}$$

$$B_{7/2} = -\frac{47\pi^{3/2}}{20160R^4}$$

$$B_4 = -\frac{202816\pi}{72747675R^5}$$

$$B_{9/2} = -\frac{521\pi^{3/2}}{443520R^6}$$

$$B_5 = -\frac{25426048\pi}{15058768725R^7}$$

$$B_{11/2} = -\frac{9521\pi^{3/2}}{11531520R^8}$$

$$B_6 = -\frac{90878576896\pi}{67689165418875R^9}$$

$$B_{13/2} = -\frac{34344493\pi^{3/2}}{47048601600R^{10}}$$

$$B_7 = -\frac{22835854180352\pi}{17531493843488625R^{11}}$$

$$B_{15/2} = -\frac{36201091\pi^{3/2}}{47048601600R^{12}}$$

$$B_8 = -\frac{1509389910845640704\pi}{1019964780320324713875R^{13}}$$

$$B_{17/2} = -\frac{153984929039\pi^{3/2}}{164481911193600R^{14}}$$

$$B_9 = -\frac{16734502326056390696\pi}{8724778730860057602486875R^{15}}$$

$$B_{19/2} = -\frac{13334525091737\pi^{3/2}}{10362360405196800R^{16}}$$

$$B_{10} = -\frac{643985013732181345325056\pi}{2312066363677915264658988875R^{17}}.$$

In four dimensions the result is

$$B_0 = \frac{1}{2}\pi^2 R^4$$

$$B_{1/2} = -\pi^{5/2}R^3$$

$$B_1 = 2\pi^2 R^2$$

$$B_{3/2} = -\frac{11\pi^{5/2}R}{32}$$

$$B_2 = -\frac{4\pi^2}{45}$$

$$B_{5/2} = -\frac{35\pi^{5/2}}{4096R}$$

$$B_3 = -\frac{464\pi^2}{45045R^2}$$

$$B_{7/2} = -\frac{911\pi^{5/2}}{196608R^3}$$

$$B_4 = -\frac{107456\pi^2}{14549535R^4}$$

$$B_{9/2} = -\frac{827315\pi^{5/2}}{201326592R^5}$$

$$B_5 = -\frac{23288576\pi^2}{3011753745R^6}$$

$$B_{11/2} = -\frac{158590273\pi^{5/2}}{32212254720R^7}$$

$$B_6 = -\frac{20064545792\pi^2}{1933976154825R^8}$$

$$B_{13/2} = -\frac{630648945109\pi^{5/2}}{86586540687360R^9}$$

$$B_7 = -\frac{492912963584\pi^2}{29464695535275R^{10}}$$

$$B_{15/2} = -\frac{70309732006867\pi^{5/2}}{55415386039910040R^{11}}$$

$$B_8 = -\frac{37648078688043008\pi^2}{1204208713483264125R^{12}}$$

$$B_{17/2} = -\frac{1578924180477650401\pi^{5/2}}{62419890835355074560R^{13}}$$

$$B_9 = -\frac{887504373820227584\pi^2}{13409327181833639595R^{14}}$$

$$B_{19/2} = -\frac{1018264365864160946171\pi^{5/2}}{1797692856058226147 3280R^{15}}$$

$$B_{10} = -\frac{25262955115582847949 2096\pi^2}{161682962494959109416712 5R^{16}}.$$

Finally, in five dimensions we obtain

$$B_0 = \frac{8\pi^2 R^5}{15}$$

$$B_{1/2} = -\frac{4\pi^{5/2}R^4}{3}$$

$$B_1 = \frac{32\pi^2 R^3}{9}$$

$$B_{3/2} = -\pi^{5/2}R^2$$

$$B_2 = -\frac{128\pi^2 R}{945}$$

$$B_{5/2} = \frac{17\pi^{5/2}}{360}$$

$$B_3 = \frac{1216\pi^2}{45045R}$$

$$B_{7/2} = \frac{157\pi^{5/2}}{30240R^2}$$

$$B_4 = \frac{235264\pi^2}{43648605R^3}$$

$$B_{9/2} = \frac{5\pi^{5/2}}{2464R^4}$$

$$B_5 = \frac{779264\pi^2}{280598175R^5}$$

$$B_{11/2} = \frac{593\pi^{5/2}}{449280R^6}$$

$$B_6 = \frac{91757946368\pi^2}{43074923448375R^7}$$

$$B_{13/2} = \frac{32815499\pi^{5/2}}{28229160960R^8}$$

$$B_7 = \frac{22103738934272\pi^2}{10518896306093175R^9}$$

$$B_{15/2} = \frac{119034319\pi^{5/2}}{94097203200R^{10}}$$

$$B_8 = \frac{53300366610079744\pi^2}{21397862524202616375R^{11}}$$

$$B_{17/2} = \frac{798608979601\pi^{5/2}}{493445733580800R^{12}}$$

$$B_9 = \frac{381809787573414866944\pi^2}{111856137575128943621625R^{13}}$$

$$B_{19/2} = \frac{146801666871373\pi^{5/2}}{62174162431180800R^{14}}$$

$$B_{10} = \frac{31815282789579439112192\pi^2}{60314774704641267773712755R^{15}}$$

C Heat-kernel coefficients for Neumann boundary conditions

Here is a list of the results we have obtained for the heat-kernel coefficients of the Laplace operator in 3, 4 and 5 dimensions with Neumann boundary conditions. In three dimensions we have found

$$B_0 = \frac{4}{3}\pi R^3$$

$$B_{1/2} = 2\pi^{3/2}R^2$$

$$B_1 = \frac{8\pi R}{3}$$

$$B_{3/2} = \frac{7}{6}\pi^{3/2}$$

$$B_2 = \frac{16\pi}{9R}$$

$$B_{5/2} = \frac{47\pi^{3/2}}{60R^2}$$

$$B_3 = \frac{6464\pi}{6435R^3}$$

$$B_{7/2} = \frac{3973\pi^{3/2}}{10080R^4}$$

$$B_4 = \frac{14766656\pi}{31177575R^5}$$

$$B_{9/2} = \frac{5057\pi^{3/2}}{28160R^6}$$

$$B_5 = \frac{2314167424\pi}{10756263375R^7}$$

$$B_{11/2} = \frac{2320069\pi^{3/2}}{27675648R^8}$$

$$B_6 = \frac{1439468204288\pi}{13537833083775R^9}$$

$$B_{13/2} = \frac{11298472831\pi^{3/2}}{250925875200R^{10}}$$

$$B_7 = \frac{369968178163712\pi}{5843831281162875R^{11}}$$

$$B_{15/2} = \frac{1718717967893\pi^{3/2}}{57211099545600R^{12}}$$

$$B_8 = \frac{48366532825354366976\pi}{10199647803203247 13875R^{13}}$$

$$B_{17/2} = \frac{113384991528329\pi^{3/2}}{4511503849881600R^{14}}$$

$$B_9 = \frac{78198023712592304 5376\pi}{178056708793062400 05075R^{15}}$$

$$B_{19/2} = \frac{33839928581307889\pi^{3/2}}{1326382131865190400R^{16}}$$

$$B_{10} = \frac{14392436216775440050663424\pi}{29726567533001 7676884727125R^{17}}$$

In four dimensions

$$B_0 = \frac{1}{2}\pi^2 R^4$$

21

$$
\begin{aligned}
B_{1/2} &= \pi^{5/2} R^3 \\
B_1 &= 2\pi^2 R^2 \\
B_{3/2} &= \frac{41\pi^{5/2} R}{32} \\
B_2 &= \frac{116\pi^2}{45} \\
B_{5/2} &= \frac{5861\pi^{5/2}}{4096 R} \\
B_3 &= \frac{99472\pi^2}{45045 R^2} \\
B_{7/2} &= \frac{388657\pi^{5/2}}{393216 R^3} \\
B_4 &= \frac{18334144\pi^2}{14549535 R^4} \\
B_{9/2} &= \frac{91095533\pi^{5/2}}{201326592 R^5} \\
B_5 &= \frac{6269294336\pi^2}{15058768725 R^6} \\
B_{11/2} &= \frac{2096614963\pi^{5/2}}{32212254720 R^7} \\
B_6 &= -\frac{1448614636544\pi^2}{13537833083775 R^8} \\
B_{13/2} &= -\frac{13041149176631\pi^{5/2}}{86586540687360 R^9} \\
B_7 &= -\frac{38509398708224\pi^2}{100179964819935 R^{10}} \\
B_{15/2} &= -\frac{14987877600061463\pi^{5/2}}{5541538603991040 R^{11}} \\
B_8 &= -\frac{75623979333176688864\pi^2}{13246295848315905375 R^{12}} \\
B_{17/2} &= -\frac{23865356170241004641\pi^{5/2}}{62419890835355074560 R^{13}} \\
B_9 &= -\frac{30045051913611575296\pi^2}{36622112051226326625 R^{14}} \\
B_{19/2} &= -\frac{135252966433194092697787\pi^{5/2}}{233700071287569399152640 R^{15}} \\
B_{10} &= -\frac{30784375321962136705 4336\pi^2}{230975660707084442023875 R^{16}}
\end{aligned}
$$

And, finally, in five dimensions

$$
B_0 = \frac{8\pi^2 R^5}{15}
$$

$$B_{1/2} = \frac{4\pi^{5/2}R^4}{3}$$

$$B_1 = \frac{32\pi^2 R^3}{9}$$

$$B_{3/2} = 3\pi^{5/2}R^2$$

$$B_2 = \frac{1024\pi^2 R}{135}$$

$$B_{5/2} = \frac{1873\pi^{5/2}}{360}$$

$$B_3 = \frac{63296\pi^2}{6435R}$$

$$B_{7/2} = \frac{10121\pi^{5/2}}{1890R^2}$$

$$B_4 = \frac{504064\pi^2}{61047R^3}$$

$$B_{9/2} = \frac{198463\pi^{5/2}}{55440R^4}$$

$$B_5 = \frac{125689856\pi^2}{30879225R^5}$$

$$B_{11/2} = \frac{34154807\pi^{5/2}}{34594560R^6}$$

$$B_6 = -\frac{56447170574848\pi^2}{157941385977375R^7}$$

$$B_{13/2} = -\frac{16602940093\pi^{5/2}}{14114580480R^8}$$

$$B_7 = -\frac{945576485184512\pi^2}{281253911927625R^9}$$

$$B_{15/2} = -\frac{13550828636809\pi^{5/2}}{5721109954560R^{10}}$$

$$B_8 = -\frac{259104011527854628864\pi^2}{55634442562926802575R^{11}}$$

$$B_{17/2} = -\frac{5379580705269259\pi^{5/2}}{1973782934323200R^{12}}$$

$$B_9 = -\frac{4618067750093566203084 8\pi^2}{95876689350110523104 25R^{13}}$$

$$B_{19/2} = -\frac{2640354677256557617\pi^{5/2}}{994786598898892800R^{14}}$$

$$B_{10} = -\frac{1401638457879249954799616\pi^2}{30677572273479636417412 5R^{15}}$$

D Heat-kernel coefficients for Robin boundary conditions

We conclude our list of results with the leading coefficients for general Robin boundary conditions for $D = 3, 4$ and 5.

In three dimensions, we have found

$$B_0 = \frac{4\pi R^3}{3}$$

$$B_{1/2} = 2\pi^{3/2} R^2$$

$$B_1 = -\frac{4\pi R}{3}(1 + 6u)$$

$$B_{3/2} = \frac{\pi^{3/2}}{6}(1 + 24u^2)$$

$$B_2 = \frac{2\pi}{45R}(1 - 18u + 60u^2 - 120u^3)$$

$$B_{5/2} = \frac{\pi^{3/2}}{60R^2}(2 - 15u + 60u^2 - 120u^3 + 120u^4)$$

$$B_3 = \frac{\pi}{45045R^3}(1633 - 12870u + 46904u^2 \\ - 107536u^3 + 144144u^4 - 96096u^5)$$

$$B_{7/2} = \frac{\pi^{3/2}}{10080R^4}(151 - 1008u + 3612u^2 \\ - 8400u^3 + 13440u^4 - 13440u^5 + 6720u^6)$$

$$B_4 = \frac{\pi}{436486050R^5}(8243319 - 51363270u + 169826940u^2 \\ - 395830040u^3 + 676878800u^4 - 835097120u^5 \\ + 665121600u^6 - 266048640u^7)$$

$$B_{9/2} = \frac{\pi^{3/2}}{1774080R^6}(14639 - 80784u + 249304u^2 \\ - 556600u^3 + 976800u^4 - 1330560u^5 \\ + 1340416u^6 - 887040u^7 + 295680u^8)$$

$$B_5 = \frac{\pi}{301175374500R^7}(3517532467 - 17760354570u \\ + 49945523040u^2 - 105573378240u^3 + 182023225440u^4 \\ - 259648898880u^5 + 295543449600u^6 - 252181862400u^7 \\ + 142779436800u^8 - 40794124800u^9)$$

In four dimensions, the results read

$$B_0 = \frac{\pi^2 R^4}{2}$$

$$B_{1/2} = \pi^{5/2}R^3$$

$$B_1 = -2\pi^2 R^2(1 + 2u)$$

$$B_{3/2} = \frac{\pi^{5/2}R}{32}(9 + 32u + 64u^2)$$

$$B_2 = -\frac{4\pi^2}{45}(1 + 30u^3)$$

$$B_{5/2} = -\frac{\pi^{5/2}}{4096R}(59 - 224u + 2048u^3 - 4096u^4)$$

$$B_3 = -\frac{16\pi^2}{45045R^2}(75 - 286u + 286u^2 + 858u^3 \\ -3003u^4 + 3003u^5)$$

$$B_{7/2} = -\frac{\pi^{5/2}}{393216R^3}(5807 - 21024u + 29952u^2 \\ +7168u^3 - 110592u^4 + 196608u^5 - 131072u^6)$$

$$B_4 = -\frac{32\pi^2}{14549535R^4}(11726 - 39368u + 62016u^2 \\ -36176u^3 - 75582u^4 + 230945u^5 \\ -277134u^6 + 138567u^7)$$

$$B_{9/2} = -\frac{\pi^{5/2}}{201326592R^5}(2961171 - 9105152u + 14440448u^2 \\ -13142016u^3 - 458752u^4 + 25427968u^5 \\ -46137344u^6 + 41943040u^7 - 16777216u^8)$$

$$B_5 = -\frac{64\pi^2}{15058768725R^6}(6419236 - 17976600u + 27448200u^2 \\ -28336920u^3 + 14866740u^4 + 14709420u^5 \\ -49365705u^6 + 65189475u^7 - 47805615u^8 \\ +15935205u^9)$$

Finally, in five dimensions we have found

$$B_0 = \frac{8\pi^2 R^5}{15}$$

$$B_{1/2} = \frac{4\pi^{5/2}R^4}{3}$$

$$B_1 = -\frac{8\pi^2 R^3}{9}(5 + 6u)$$

$$B_{3/2} = \frac{\pi^{5/2}R^2}{3}(3 + 8u + 8u^2)$$

$$B_2 = \frac{4\pi^2 R}{135}(-5 + 6u - 60u^2 - 120u^3)$$

$$B_{5/2} = \frac{\pi^{5/2}}{360}(-17 - 240u^2 + 480u^4)$$

$$B_3 = \frac{2\pi^2}{135135R}(87 + 13442u - 35464u^2 + 61776u^3 + 48048u^4 - 96096u^5)$$

$$B_{7/2} = \frac{\pi^{5/2}}{7560R^2}(-88 + 483u - 1806u^2 + 2940u^3 - 1680u^4 - 3360u^5 + 3360u^6)$$

$$B_4 = \frac{\pi^2}{43648605R^3}(-539501 + 4050078u - 12086660u^2 + 23878744u^3 - 23715952u^4 + 1478048u^5 + 26604864u^6 - 17736576u^7)$$

$$B_{9/2} = \frac{\pi^{5/2}}{2661120R^4}(-18927 + 99616u - 302720u^2 + 576048u^3 - 748704u^4 + 473088u^5 + 177408u^6 - 591360u^7 + 295680u^8)$$

$$B_5 = \frac{\pi^2}{90352612350R^5}(-935536567 + 4964319990u - 13111462800u^2 + 25019918880u^3 - 34365190560u^4 + 32451298368u^5 - 12409401600u^6 - 12609093120u^7 + 20397062400u^8 - 8158824960u^9).$$

This concludes our lists of explicit tables for the heat-kernel coefficients. In the same way, results for any desired dimension D are very easy to obtain from the formulas in the text.

References

[1] Gilkey P B. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* Publish or Perish, Vilmington (1984).

[2] Birrell N and Davies P C W. *Quantum Fields in Curved Spaces.* Cambridge University Press, Cambridge (1982).

[3] De Witt B S. Phys. Rep. C, **19**, 295 (1975).

[4] Hawking S W. Commun. Math. Phys., **55**, 133 (1977).

[5] Blau S K, Visser M, and Wipf A. Nucl. Phys. B, **310**, 163 (1988).

[6] Dolan B P and Nash C. Commun. Math. Phys., **148**, 139 (1992).

[7] Greiner P. Arch. Rat. Mech. Anal., **41**, 163 (1971).

[8] Gilkey P B. J. Diff. Geom., **10**, 601 (1976).

[9] Avramidi I G. Nucl. Phys. B, **355**, 712 (1991).

[10] Fulling S A and Kennedy G. Transac. Amer. Math. Soc., **310**, 583 (1988).

[11] Amsterdamski P, Berkin A L, and O'Connor D J. Class. Quantum Grav., **6**, 1981 (1989).

[12] Branson T P and Gilkey P B. Commun. PDE, **15**, 245 (1990).

[13] Moss I G and Dowker J S. Phys. Lett. B, **229**, 261 (1989).

[14] Mc Avity D M and Osborn H. Class. Quantum Grav., **8**, 603 (1991).

[15] Dettki A and Wipf A. Nucl. Phys. B, **377**, 252 (1992).

[16] Dowker J S and Schofield J P. J. Math. Phys., **31**, 808 (1990).

[17] Cognola G, Vanzo L, and Zerbini S. Phys. Lett. B, **241**, 381 (1990).

[18] Branson T P and Gilkey P B. *The asymptotics of the Laplacian on a manifold with boundary II*, in preparation.

[19] Moss I G and Poletti S. Nucl. Phys. B, **341**, 155 (1990).

[20] Moss I G and Poletti S. Phys. Lett. B, **245**, 355 (1990).

[21] D'Eath P D and Esposito G. Phys. Rev. D, **43**, 3234 (1991).

[22] Esposito G. Class. Quantum Grav., **11**, 905 (1994).

[23] Esposito G, Kamenshchik A Yu, Mishakov I V, and Pollifrone G. Class. Quantum Grav., **11**, 2939 (1994).

[24] Esposito G, Kamenshchik A Yu, Mishakov I V, and Pollifrone G. Phys. Rev. D, **50**, 6329 (1994).

[25] Esposito G. *Quantum gravity, quantum cosmology and Lorentzian geometries.* Second corrected and enlarged edition, Lecture Notes in Physics, New Series m: Monographs, Vol. m12, Springer-Verlag, Berlin (1994).

[26] Kamenshchik A Yu and Mishakov I V. Int. J. Mod. Phys. A, **7**, 3713 (1992).

[27] Barvinsky A O, Kamenshchik A Yu, and Karmazin I P. Ann. Phys., **219**, 201 (1992).

[28] Vassilevich D V. J. Math. Phys., to appear (1995).

[29] Moss I G and Poletti S. Phys. Lett. B, **333**, 326 (1994).

[30] Voros A. Commun. Math., **110**, 439 (1987).

[31] Elizalde E, Odintsov S D, Romeo A, Bytsenko A A, and Zerbini S. *Zeta regularization techniques with applications.* World Sci., Singapore (1994).

[32] Stewartson K and Waechter R T. Proc. Camb. Phil. Soc., **69**, 355 (1971).

[33] Waechter R T. Proc. Camb. Phil. Soc., **72**, 439 (1972).

[34] Kennedy G. PhD tesis, Manchester (1978).

[35] Kennedy G. J. Phys. A: Math. Gen., **11**, L173 (1978).

[36] Dowker J S and Apps J S. *Further functional determinants*, Preprint MUTP/95/1, The University of Manchester.

[37] Erdelyi A, Magnus W, Oberhettinger F, and Tricomi F G. *Higher transcendental functions Vol 1.* McGraw-Hill Book Company, New York (1953).

[38] Kamenshchik A Yu and Karmazin I P. Int. J. Mod. Phys. A, **7**, 3713 (1992).

[39] Bordag M. J. Phys. A: Math. Gen., to appear (1995).

[40] Elizalde E, Leseduarte S, and Romeo A. J. Phys. A: Math. Gen., **26**, 2409 (1993).

[41] Leseduarte S and Romeo A. J. Phys. A: Math. Gen., **27**, 2483 (1994).

[42] Abramowitz M and Stegun I A. *Handbook of Mathematical Functions (Natl. Bur. Stand. Appl. Math. Ser.55).* (Washington, D.C.: US GPO), New York: Dover (reprinted 1972).

[43] Gradshteyn I S and Ryzhik I M. *Tables of Integrals, Series and Products.* Academic Press, New York (1965).

[44] Weldon H A. Nucl. Phys. B, **270**, 79 (1986).

[45] Elizalde E and Romeo A. Phys. Rev. D, **40**, 436 (1989).

[46] Elizalde E, Kirsten K, and Zerbini S. J. Phys. A: Math. Gen., **28**, 617 (1995).

[47] Levitin M, private communication.